Computational Design of a Kit of Parts for Bending-Active Structures

Supplementary Material

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This document provides additional technical details regarding our process for calculating gradients for the preservation energy term $\mathcal{F}$, optimization objective non-dimensionalization, and bound constraints guarantees on the parts. We also show the sets of designs we used for the experiments in the main paper.

1 Preservation Energy Optimization

Our preservation energy $\mathcal{F}(\mathbf{x}, \mathbf{r})$ is defined for a physical system with rest variables $\mathbf{r}$ and simulation variables $\mathbf{x}$ as the sum of a target fitting term $\mathcal{T}(\mathbf{x})$ and an elastic energy term $\mathcal{E}(\mathbf{x}, \mathbf{r})$. We aim to efficiently compute the gradient of the preservation energy with respect to the design variables $\mathbf{d}$ (composed of rest variables and Dirichlet constraint values) when evaluated at the equilibrium state $\mathbf{x}^*(\mathbf{d})$ i.e., the gradient of $\tilde{\mathcal{F}}(\mathbf{d}) := \mathcal{F}(\mathbf{x}^*(\mathbf{d}), \mathbf{r})$.

1.1 Equilibrium Sensitivities

This term is only evaluated at the equilibrium state $\mathbf{x}^*(\mathbf{d})$ defined as We split the design variables into rest quantities e.g., beam rest lengths, and fixed deformed variables e.g., corner angles, as $\mathbf{d} = [\mathbf{r}_v^\top, \mathbf{r}_f^\top]^\top$.

$$\begin{aligned} \mathbf{x}^*(\mathbf{d}) := \underset{\mathbf{x}}{\operatorname{argmin}}\, & \bar{\mathcal{E}}(\mathbf{x}, \mathbf{r}_v) \\ \text{s.t.} \quad & \mathbf{x}_f = \mathbf{r}_f, \end{aligned} \tag{A1}$$

where $\bar{\mathcal{E}}(\mathbf{x}, \mathbf{r}_v) = \mathcal{E}(\mathbf{x}, \mathbf{r}_v) + \mathcal{D}(\mathbf{x})$ is the total energy of the system, $\mathcal{D}(\mathbf{x})$ models the external deployment forces, and $\mathbf{x}_f$ are the simulation variables fixed by the Dirichlet constraints $\mathbf{r}_f$. We express the first-order KKT conditions for the equilibrium state as

$$\begin{cases} \left.\dfrac{\partial \bar{\mathcal{E}}}{\partial \mathbf{x}}\right|_{\mathbf{x}^*(\mathbf{d}), \mathbf{d}} + \lambda(\mathbf{d})^\top \dfrac{\partial \mathbf{x}_f}{\partial \mathbf{x}} = \mathbf{0} \\ \mathbf{x}_f^*(\mathbf{d}) = \mathbf{r}_f \end{cases}$$

where λ is the Lagrange multiplier associated with the Dirichlet constraints. We further split the simulation variables into free and fixed components as $\mathbf{x} = [\mathbf{x}_v^\top, \mathbf{x}_f^\top]^\top$. We may remark that $\frac{\partial \mathbf{x}_f}{\partial \mathbf{x}}$ is a block-diagonal matrix with blocks of zeros and identity matrices, and we differentiate the equilibrium state with respect to the design variables $\mathbf{d}$ as

$$\begin{cases} \mathbf{H}_v \dfrac{\partial \mathbf{x}_v^*}{\partial \mathbf{d}} = -\begin{bmatrix} \dfrac{\partial^2 \bar{\mathcal{E}}}{\partial \mathbf{x}_v \partial \mathbf{r}_v} & \dfrac{\partial^2 \bar{\mathcal{E}}}{\partial \mathbf{x}_v \partial \mathbf{x}_f} \end{bmatrix} \\ \dfrac{\partial^2 \bar{\mathcal{E}}}{\partial \mathbf{x}_f \partial \mathbf{x}_v} \dfrac{\partial \mathbf{x}_v^*}{\partial \mathbf{d}} = -\begin{bmatrix} \dfrac{\partial^2 \bar{\mathcal{E}}}{\partial \mathbf{x}_f \partial \mathbf{r}_v} & \dfrac{\partial^2 \bar{\mathcal{E}}}{\partial {\mathbf{x}_f}^2} \end{bmatrix} - \dfrac{\partial \lambda}{\partial \mathbf{d}} \\ \dfrac{\partial \mathbf{x}_f^*}{\partial \mathbf{d}} = \begin{bmatrix} \dfrac{\partial \mathbf{x}_f^*}{\partial \mathbf{r}_v} & \dfrac{\partial \mathbf{x}_f^*}{\partial \mathbf{r}_f} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}, \end{cases}$$

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where $\mathbf{I}$ is the identity matrix, and $\mathbf{H}_v$ is the Hessian of the total energy with respect to the free simulation variables $\mathbf{x}_v$ evaluated at the equilibrium state. The sensitivity of the equilibrium state $\delta\mathbf{x}^*$ to some design variable perturbation $\delta\mathbf{d}$ can be computed as

$$\delta\mathbf{x}^* = \frac{\partial \mathbf{x}^*}{\partial \mathbf{d}}\delta\mathbf{d} = \begin{bmatrix} -\mathbf{H}_v^{-1}\left(\dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_v}\delta\mathbf{r}_v + \dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{x}_f}\delta\mathbf{r}_f\right) \\ \delta\mathbf{r}_f \end{bmatrix}.$$

Computing the second order perturbation of the equilibrium state given a perturbation of the design variables $\delta\mathbf{d}$ is done by solving the following linear system

$$\begin{cases} \mathbf{H}_v\delta\mathbf{d}\dfrac{\partial^2\mathbf{x}_v^*}{\partial\mathbf{d}^2}\delta\mathbf{d} &= -\left(\dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_v\partial\mathbf{r}_v}\delta\mathbf{r}_v + \dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_v\partial\mathbf{r}_f}\delta\mathbf{r}_f\right)\delta\mathbf{r}_v - \left(\dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_f\partial\mathbf{r}_v}\delta\mathbf{r}_v + \dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_f\partial\mathbf{r}_f}\delta\mathbf{r}_f\right)\delta\mathbf{r}_f \\ \delta\mathbf{d}\dfrac{\partial^2\mathbf{x}_f^*}{\partial\mathbf{d}^2}\delta\mathbf{d} &= 0. \end{cases}$$

1.2 Gradients

Using the chain rule, we can compute the gradient of the preservation energy with respect to the design variables as

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\mathbf{d}} = \frac{\partial\mathcal{F}}{\partial\mathbf{x}_v}\frac{\partial\mathbf{x}_v^*}{\partial\mathbf{d}} + \frac{\partial\mathcal{F}}{\partial\mathbf{x}_f}\frac{\partial\mathbf{x}_f^*}{\partial\mathbf{d}} + \frac{\partial\mathcal{F}}{\partial\mathbf{d}} = -\frac{\partial\mathcal{F}}{\partial\mathbf{x}_v}\mathbf{H}_v^{-1}\begin{bmatrix}\dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_v} & \dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{x}_f}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \dfrac{\partial\mathcal{F}}{\partial\mathbf{x}_f}\end{bmatrix} + \frac{\partial\mathcal{F}}{\partial\mathbf{d}},$$

We consequently define the adjoint state vector $\mathbf{w}$ as

$$\mathbf{H}_v\mathbf{w} = \frac{\partial\mathcal{F}}{\partial\mathbf{x}_v}^\top,$$

where the gradient of the preservation energy can be computed analytically from the own physical system's inverse design optimization pipeline. The gradient of $\tilde{\mathcal{F}}$ can thus be computed efficiently using the adjoint state vector $\mathbf{w}$ as

$$\frac{\partial\tilde{\mathcal{F}}}{\partial\mathbf{d}} = -\mathbf{w}^\top\begin{bmatrix}\dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{r}_v} & \dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{x}_f}\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \dfrac{\partial\mathcal{F}}{\partial\mathbf{x}_f}\end{bmatrix} + \frac{\partial\mathcal{F}}{\partial\mathbf{d}}.$$

1.3 Hessian Vector Products

The Hessian Vector Product (HVP) of the preservation energy given a perturbation of the design variables $\delta\mathbf{d}$ can be computed as

$$\begin{aligned}\frac{\partial^2\tilde{\mathcal{F}}}{\partial\mathbf{d}^2}\delta\mathbf{d} = &-\begin{bmatrix}\dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{r}_v\partial\mathbf{x}_v} \\ \dfrac{\partial^2\bar{\mathcal{E}}}{\partial\mathbf{x}_f\partial\mathbf{x}_v}\end{bmatrix}\delta\mathbf{w} - \begin{bmatrix}\dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{r}_v\partial\mathbf{x}_v\partial\mathbf{x}}\delta\mathbf{x}^* + \dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{r}_v\partial\mathbf{x}_v\partial\mathbf{r}_v}\delta\mathbf{r}_v \\ \dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_f\partial\mathbf{x}_v\partial\mathbf{x}}\delta\mathbf{x}^* + \dfrac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_f\partial\mathbf{x}_v\partial\mathbf{r}_v}\delta\mathbf{r}_v\end{bmatrix}\mathbf{w} \\ &+ \begin{bmatrix}\mathbf{0} \\ \dfrac{\partial^2\mathcal{F}}{\partial\mathbf{x}_f\partial\mathbf{x}}\delta\mathbf{x}^* + \dfrac{\partial^2\mathcal{F}}{\partial\mathbf{x}_f\partial\mathbf{r}_v}\delta\mathbf{r}_v\end{bmatrix} + \frac{\partial^2\mathcal{F}}{\partial\mathbf{d}\partial\mathbf{x}}\delta\mathbf{x}^* + \frac{\partial^2\mathcal{F}}{\partial\mathbf{d}^2}\delta\mathbf{d},\end{aligned}$$

where $\delta\mathbf{w}$ is the adjoint state vector perturbation, and satisfies the linear system

$$\mathbf{H}_v\delta\mathbf{w} = \frac{\partial^2\mathcal{F}}{\partial\mathbf{x}_v\partial\mathbf{x}}\delta\mathbf{x}^* + \frac{\partial^2\mathcal{F}}{\partial\mathbf{x}_v\partial\mathbf{d}}\delta\mathbf{d} - \left(\frac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{x}_v\partial\mathbf{x}}\delta\mathbf{x}^* + \frac{\partial^3\bar{\mathcal{E}}}{\partial\mathbf{x}_v\partial\mathbf{x}_v\partial\mathbf{r}_v}\delta\mathbf{r}_v\right)\mathbf{w}.$$

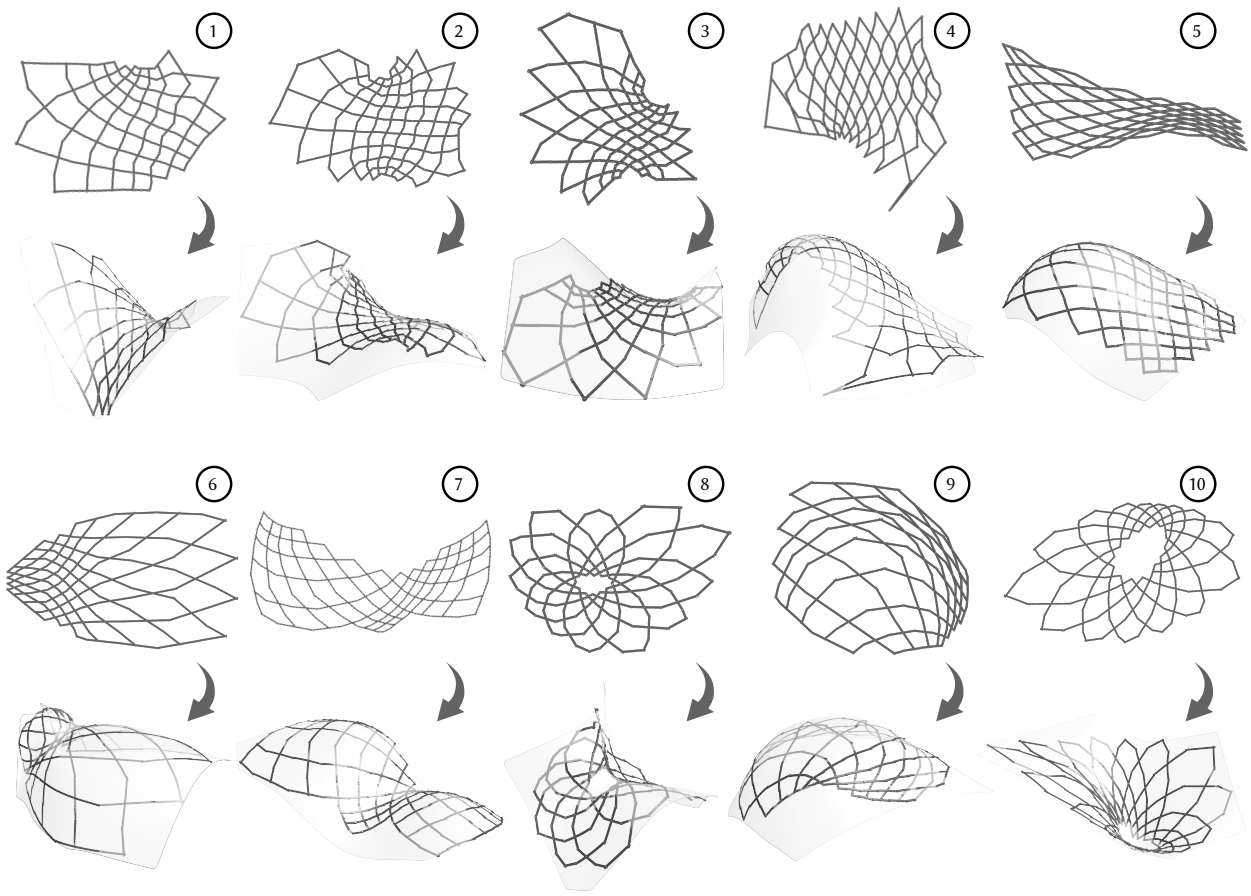

Figure 1: Conformals Set.

2 Non-Dimensionalization of the Preservation Energy

Our preservation energy $\mathcal{F}$ is the sum of the elastic energy of a deformed system $\mathcal{E}$ and a target fitting term $\mathcal{T}$. We non-dimensionalize the elastic energy by scaling it with the Young's modulus Y of the fabrication material times the rest volume V_0 of the system. The bending, twisting, and stretching energies stored in the rods of a system can be derived by plugging strain tensor fields $\boldsymbol{\epsilon}$ induced these deformation modes into the linear elasticity energy $\frac{1}{2}\int_\Omega \boldsymbol{\epsilon} : \mathbf{C} : \boldsymbol{\epsilon}$, where C is the fabrication material's elasticity tensor. For an isotropic material, $\mathbf{C} = Y\mathbf{C_0}(\nu)$, where $\mathbf{C_0}(\nu)$ depends only on the Poisson's ratio ν; for example $\mathbf{C_0}(0)$ is the fourth-order identity tensor. Since strain $\boldsymbol{\epsilon}$ is non-dimensional, this elastic energy clearly is proportional to YV_0. The target fitting term is non-dimensionalized by dividing it by the square of the bounding box diagonal of the target surface.

3 Bound Constraints Guarantees

Parts within a kit of parts may have to satisfy certain shared fabrication constraints e.g., minimum distance between corners and joints for linear elements of rationalized Orthogonal Grids and C-shells. We assume that such constraints can be expressed as bound constraints on the parts parameters, so that a part i must satisfy $\mathbf{p}_{\min} \leq \mathbf{p}_i \leq \mathbf{p}_{\max}$. We aim to show that any part $\mathbf{p}_i$ obtained from the optimal elements $\mathbf{q}^*$ using

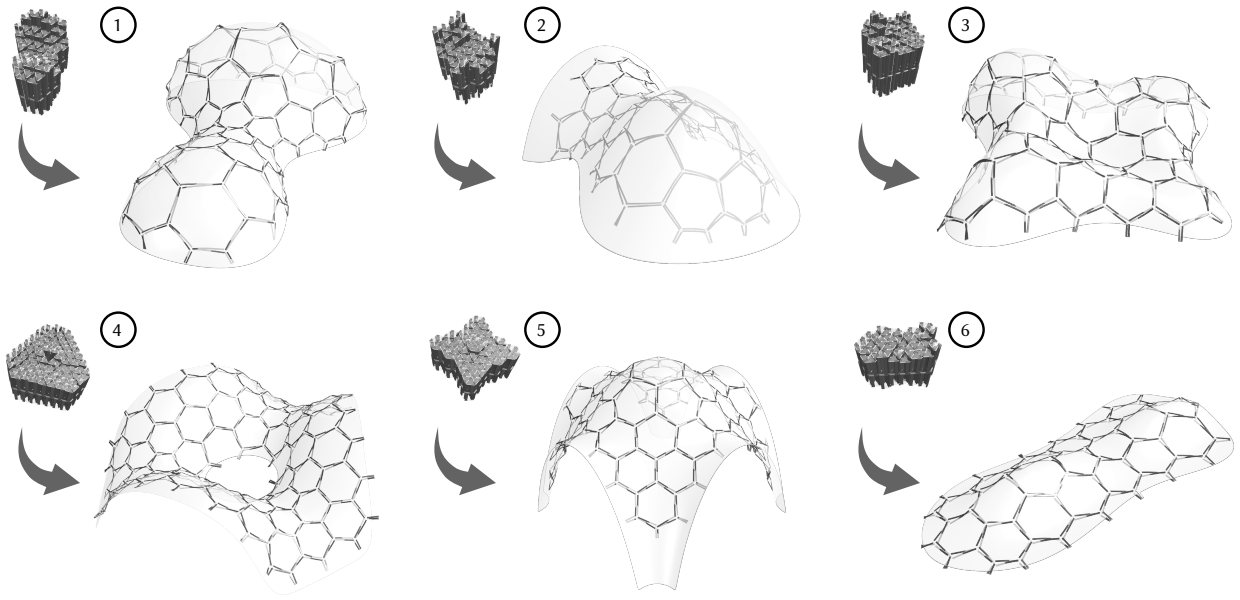

Figure 2: Umbrella Meshes Set.

the update rule

$$\mathbf{p}_i(\mathbf{q}^*) := \underset{\mathbf{y}}{\operatorname{argmin}} \sum_{j \in c^{-1}(\{i\})} \rho(\mathbf{y}, \mathbf{q}_j^*), \tag{A2}$$

is guaranteed to satisfy the original feasibility constraints for any assignment $\mathbf{c}$, provided that elements are bound during the optimization process, for the ρ we use in our work.

We impose that elements are bound by the same constraints as the parts, so that $\mathbf{p}_{\min} \leq \mathbf{q}_j \leq \mathbf{p}_{\max}$ for any element j. Under the appropriate symmetry transformations (mirroring), our projection energy ρ is a squared L^2 distance. The solution to the update rule $\mathbf{p}_i(\mathbf{q}^*)$ is given by the average of the elements assigned to the part. Therefore, the part belongs to the convex hull formed by the elements assigned to it, which itself is included in the axis-aligned bounding box formed by the corners $\mathbf{p}_{\min}$ and $\mathbf{p}_{\max}$. The part $\mathbf{p}_i(\mathbf{q}^*)$ is thus guaranteed to satisfy the original feasibility constraints. Note that unassigned parts can trivially be set to the mean of the corners $\mathbf{p}_{\min}$ and $\mathbf{p}_{\max}$ by default.

4 Datasets

We now show the sets of designs we used for the experiments in the main paper.

4.1 Conformals

The 10 designs of the Conformals dataset shown in Figure 1 are obtained following the methodology presented in Becker et al. (2024). The designs are conformal deformations of a base grid with an annulus (8 and 10) or a regular (the others) topology. Layout variations are obtained by applying a conformal map defined by a boundary a user can manipulate as presented in Eck et al. (1995) and Sawhney and Crane (2017). The target surface is extracted by smoothly interpolating the joints of the deployed C-shell.

We reuse the same surfaces for the orthogonal grids. They are initialized by either tracing uv-isolines on the target surface and converting them into piecewise straight beams or by connecting the projections of the joints on the surface using piecewise straight beams. The designs are further optimized to better fit the

target surface using an inverse design optimization algorithm similar to the one presented in Becker et al. (2023).

4.2 Umbrella Meshes

The 6 designs of the Umbrella Meshes dataset shown in Figure 2 are obtained from known shapes that have been approximated thanks to the inverse design optimization pipeline presented in Ren et al. (2022). The structure topologies i.e., the number of cells and their connectivity, are allowed to vary across experiments.

References

Becker, Q., Suzuki, S., and Pauly, M. (2024). Interactive design of c-shells using reduced parametric families. In *Proceedings of the IASS Annual Symposium*. International Association for Shell and Spatial Structures (IASS).

Becker, Q., Suzuki, S., Ren, Y., Pellis, D., Panetta, F. J., and Pauly, M. (2023). C-shells: Deployable gridshells with curved beams. *ACM Transactions on Graphics*, 42(6):1–17.

Eck, M., DeRose, T., Duchamp, T., Hoppe, H., Lounsbery, M., and Stuetzle, W. (1995). Multiresolution analysis of arbitrary meshes. In *Proceedings of the 22nd annual conference on Computer graphics and interactive techniques*, pages 173–182.

Ren, Y., Kusupati, U., Panetta, J., Isvoranu, F., Pellis, D., Chen, T., and Pauly, M. (2022). Umbrella meshes: elastic mechanisms for freeform shape deployment. *ACM Transactions on Graphics*, 41(ARTICLE):1–15.

Sawhney, R. and Crane, K. (2017). Boundary first flattening. *ACM Transactions on Graphics (ToG)*, 37(1):1–14.